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Critical behavior of a general $O(n)$-symmetric model of two n-vector fields in $D = 4 - 2\epsilon$

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Abstract

The critical behavior of the $O(n)$-symmetric model with two n-vector fields is studied within the field-theoretical renormalization group approach in a $D = 4 - 2\epsilon$ expansion. Depending on the coupling constants, the β-functions, fixed points and critical exponents are calculated up to the one- and two-loop order, respectively (η in two- and three-loop order). Both continuous lines of fixed points and $O(n) \times O(2)$ invariant discrete solutions were found. Apart from already known fixed points two new ones arise. One agrees in one-loop order with a known fixed point, but differs from it in two-loop order.

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1. Introduction

The renormalization group approach provides a natural framework for the understanding of critical properties of phase transitions. A very large variety of critical phenomena can be described by the so-called ϕ^4 models. The simple $O(n)$-symmetric one-field model

$$S_{O(n)}(\phi) = \tfrac{1}{2}[(\nabla\phi)^2 + \tau\phi^2] + \tfrac{1}{4!}g(\phi^2)^2, \tag{1}$$

where $\phi = (\phi_1, \ldots, \phi_n)$ is a real n-component vector field, while τ is a temperature-like parameter and $g > 0$, was extended in [1] to the interplay of two vector fields under the $O(n)$+$O(m)$ symmetry

$$S_{O(n)+O(m)}(\phi_1, \phi_2) = \tfrac{1}{2}\left[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + \tau_1\phi_1^2 + \tau_2\phi_2^2\right] + \tfrac{1}{4!}\left[g_1\left(\phi_1^2\right)^2 + g_2\left(\phi_2^2\right)^2 + g_3\left(\phi_1^2\right)\left(\phi_2^2\right)\right]. \tag{2}$$

Six different fixed points were found. Three of them are always unstable and the stability of three others depends on n and m. The $O(n)$+$O(m)$ model has been used to describe multicritical phenomena. We mention the critical behavior of uniaxial antiferromagnets in a magnetic field parallel to the field direction [1] and the $SO(5)$ theory of high-T_c superconductors [2–4]. Also interesting phenomena of inverse symmetry breaking, symmetry nonrestoration and re-entrant phase transitions were reported [5, 6]. This model as well as model (3) presented below has also been investigated in [7].

Recently, frustrated spin systems with noncollinear or canted spin ordering have been the object of intensive research [8–11]. Examples are helical magnets and layered triangular Heisenberg antiferromagnets [12]. In the corresponding action

$$S_{O(n)\times O(2)}(\phi_1,\phi_2) = \tfrac{1}{2}\big[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + \tau\big(\phi_1^2+\phi_2^2\big)\big] \\ + \tfrac{1}{4!}u\big(\phi_1^2+\phi_2^2\big)^2 + \tfrac{1}{4!}v\big[(\phi_1\phi_2)^2 - \big(\phi_1^2\big)\big(\phi_2^2\big)\big] \qquad (3)$$

the scalar product $\phi_1\phi_2$ is present [13–15]. Both fields have n components and the model possesses the $O(n)\times O(2)$ symmetry. In the $4-2\epsilon$ expansion, the number of fix points (FP) and their stability depend on n, however different theoretical methods lead to contradictory results [4].

The results based on three-loop renormalization group calculations [16–18] show that in three-dimensional chiral magnets with $n = 2, 3$ critical fluctuations destroy continuous phase transitions converting them into the first-order ones, i.e. the chiral class of universality does not exist. On the other hand, the analysis of the higher-order—five-loop and six-loop—$3D$ RG expansions reveals a new stable fixed point for physical values of n [19]. This new fixed point turns out to be a focus [20] that governs the critical behavior of the system in a somewhat unusual way. It was found to exist only for $n < 6$ [21] having no generic relation to the stable chiral fixed point seen at small ϵ and large n, since for small ϵ one obtains complex ω's in the region of complex fixed point couplings, whereas the analysis [19–21] predicts real fixed point couplings and complex ω's. This indicates that such fixed points are not found by simple continuation in ϵ. The situation in two dimensions seems to be similar [22].

The major part of the results obtained within other approaches ('exact' renormalization group, Monte Carlo simulations, etc) may be considered as favoring the fluctuation-induced first-order chiral transitions for $n = 2, 3$ [23–29]. Such transitions are characterized by effective critical exponents that are non-universal and depend on the magnet or antiferromagnet studied. Arguments were presented [30] that the new chiral fixed point found in [19] may be an artifact produced by rather long RG expansions. For detailed discussion and most recent results see, e.g. [31, 32].

The purpose of this paper is to investigate the critical behavior of the general $O(n)$ symmetric theory

$$\begin{aligned} S_{O(n)}(\phi_1,\phi_2) &= S_0(\phi_1,\phi_2,\tau) + S_{\text{int}}(\phi_1,\phi_2,g),\\ S_0(\phi_1,\phi_2,\tau) &= \frac{1}{2}\left(\sum_{k=1}^{2}(\nabla\phi_k)^2 + \sum_{k=1}^{3}\tau_k\mathcal{I}_k\right),\\ S_{\text{int}}(\phi_1,\phi_2,g) &= \frac{1}{8}\sum_{k,l=1}^{3}\mathcal{I}_k g_{kl}\mathcal{I}_l = \frac{1}{8}\mathcal{I}g\mathcal{I} \end{aligned} \qquad (4)$$

of two classical fields with n components respectively, with

$$\begin{pmatrix}\mathcal{I}_1\\ \mathcal{I}_2\\ \mathcal{I}_3\end{pmatrix} = \begin{pmatrix}\phi_1^2\\ \phi_2^2\\ \sqrt{2}\phi_1\phi_2\end{pmatrix}. \qquad (5)$$

g is assumed to be symmetric, $g_{ji} = g_{ij}$. Whenever possible we use only g_{ij} with $i \leqslant j$. The model (4) becomes $O(n)+O(n)$ symmetric when $\tau_3 = g_{13} = g_{23} = g_{33} = 0$. On the other hand, setting

$$\tau_1 = \tau_2, \qquad \tau_3 = g_{13} = g_{23} = 0, \qquad g_{11} = g_{22}, \qquad g_{12} = g_{11} - g_{33} \qquad (6)$$

leads to the $O(n)\times O(2)$ model of frustrated spins.

As a function of n we find 10 FPs in total. To our knowledge the FPs we denote by *RS 2.1b* and *RS 2.3* are new ones. The FP *RS 2.1b* is remarkable, since in one-loop order it coincides with the FP *RS 2.1a*, which describes two decoupled isotropic systems. *RS 2.1b* shows in order $\epsilon^{3/2}$ a coupling between both systems for general n.

In the following section we give the expression of the β-function and of various anomalous dimensions, which allow the determination of the critical exponents η, ν and ω and the cross-over exponents in one-loop order for model (4). In section 3, we consider orthogonal transformations between the two fields ϕ_1 and ϕ_2. As a consequence there will be discrete FPs (invariant under this transformation) and lines of FPs. Then we classify the solutions according to the behavior in the large-n limit. In section 5, the various fixed points are determined and the corresponding critical exponents are given for finite n. If in some range of n the FP becomes complex, we determine in order ϵ the limit n_c, where it becomes complex (for positive n only). Comparison is made with the known models (1)–(3) in section 6. A summary concludes the paper.

2. The 4 − 2ϵ expansion

The expression for the critical exponents can be taken from the review article by Brézin *et al* [33]. Writing

$$S_{\text{int}} = \tfrac{1}{4!} g_{i\alpha,j\beta,k\kappa,l\lambda}\phi_{i\alpha}\phi_{j\beta}\phi_{k\kappa}\phi_{l\lambda} \tag{7}$$

one obtains

$$g_{i\alpha,j\beta,k\kappa,l\lambda} = \frac{1}{\upsilon_{i,j}\upsilon_{k,l}} g_{\rho_{i,j},\rho_{k,l}}\delta_{\alpha\beta}\delta_{\kappa,\lambda} + \frac{1}{\upsilon_{i,k}\upsilon_{j,l}} g_{\rho_{i,k},\rho_{j,l}}\delta_{\alpha\kappa}\delta_{\beta\lambda} + \frac{1}{\upsilon_{i,l}\upsilon_{j,k}} g_{\rho_{i,l},\rho_{j,k}}\delta_{\alpha\lambda}\delta_{\beta\kappa}, \tag{8}$$

$$\begin{aligned} &\upsilon_{1,1} = \upsilon_{2,2} = 1, \qquad \upsilon_{1,2} = \upsilon_{2,1} = \sqrt{2}, \qquad \rho_{1,1} = 1, \qquad \rho_{2,2} = 2, \\ &\rho_{1,2} = \rho_{2,1} = 3. \end{aligned} \tag{9}$$

The six β functions $\beta_{ij} \equiv \mu\partial_\mu g_{ij}$, where μ is an auxiliar parameter with the critical dimension 1, can be written in one-loop order

$$\beta_{ij} = -2\epsilon g_{ij} + \tfrac{1}{2}(n+8) g_{ik} g_{kl} + \tfrac{1}{2} C_{ij,kl,mn} g_{kl} g_{mn} \tag{10}$$

with

$$\begin{array}{ll} i,j & C_{ij,kl,mn} g_{kl} g_{mn} \\ 1,1 & -8g_{12}^2 + 2g_{12}g_{33} + g_{33}^2 \\ 1,2 & -6g_{11}g_{12} - 6g_{12}g_{22} - 4g_{13}g_{23} + g_{11}g_{33} + 4g_{12}^2 + 2g_{13}^2 + g_{22}g_{33} + 2g_{23}^2 + g_{33}^2 \\ 1,3 & -6g_{12}g_{23} - 3g_{13}g_{33} + 6g_{12}g_{13} + 3g_{23}g_{33} \\ 2,2 & -8g_{12}^2 + 2g_{12}g_{33} + g_{33}^2 \\ 2,3 & -6g_{12}g_{13} - 3g_{23}g_{33} + 6g_{12}g_{23} + 3g_{13}g_{33} \\ 3,3 & -2g_{13}^2 - 2g_{23}^2 - 6g_{33}^2 + 2g_{11}g_{33} + 8g_{12}g_{33} + 4g_{13}g_{23} + 2g_{22}g_{33}. \end{array} \tag{11}$$

We have rescaled the couplings by a factor $8\pi^2$ as usual.

The FPs g^* are the solutions of $\beta_{ij}(g^*) = 0$. We observe that (4) is symmetric under the simultaneous interchange of g_{11} with g_{22} and g_{13} with g_{23}. The simultaneous change of signs of g_{13} and g_{23} leaves the solution to (10) invariant.

The stability matrix

$$\omega_{ij,kl} = \partial\beta_{ij}(g)/\partial g_{kl}|_{g=g^*} \tag{12}$$

is easily obtained. The eigenvalues of (12) are the critical exponents ω.

Similarly the critical exponents η are obtained from the eigenvalues γ^*_Φ of the symmetric 2×2 matrix γ_Φ at $g = g^*$,

$$\begin{aligned}
\{\gamma_\Phi\}_{11} &= \frac{1}{16}\big(2(n+2)g_{11}^2 + (n+2)g_{23}^2 + (n+1)g_{33}^2 + 2ng_{12}^2 + 4g_{12}g_{33} + 3(n+2)g_{13}^2\big),\\
\{\gamma_\Phi\}_{21} &= \frac{\sqrt{2}(n+2)}{16}\big((g_{11}+g_{12}+g_{33})g_{13} + (g_{22}+g_{12}+g_{33})g_{23}\big),\\
\{\gamma_\Phi\}_{22} &= \frac{1}{16}\big((n+2)g_{13}^2 + 2(n+2)g_{22}^2 + 2ng_{12}^2 + 3(n+2)g_{23}^2 + (n+1)g_{33}^2 + 4g_{12}g_{33}\big),
\end{aligned} \tag{13}$$

calculated at the specific FP, with respect to $\eta_i = 2\gamma^*_{\Phi_i}$.

The critical behavior of perturbations bilinear in the fields ϕ is governed by the expression for $1/\nu - 2$ given by Brézin *et al* which as a function of the n components of the fields can be written as

$$\left(\frac{1}{\nu} - 2\right)_{i\alpha,j\beta;k\kappa,l\lambda} = d^{(1)}_{i,j,k,l}\delta_{\alpha\beta}\delta_{\kappa\lambda} + d^{(2)}_{i,j,k,l}\delta_{\alpha\kappa}\delta_{\beta\lambda} + d^{(2)}_{i,j,l,k}\delta_{\alpha\lambda}\delta_{\beta\kappa}. \tag{14}$$

Eigenfunctions of this matrix are of three types:

(i) They may be $O(n)$ symmetric corresponding to the variation of the τ_i. Thus one applies eigenfunctions of type $a_{kl}\delta_{\kappa\lambda}$ to (14) and with

$$\hat{d}^{(1)}_{\rho_{i,j},\rho_{k,l}} = \upsilon_{i,j}\upsilon_{k,l}d^{(1)}_{i,j,k,l}, \qquad \hat{d}^{(2)}_{\rho_{i,j},\rho_{k,l}} = \upsilon_{i,j}\upsilon_{k,l}\big(d^{(2)}_{i,j,k,l} + d^{(2)}_{i,j,l,k}\big) \tag{15}$$

the eigenvalues are those of the 3×3 matrix

$$\gamma_\tau = n\hat{d}^{(1)} + \hat{d}^{(2)}, \tag{16}$$

which in one-loop order reads

$$\gamma_\tau = -\frac{1}{2}\begin{pmatrix} (n+2)g_{11} & ng_{12}+g_{33} & (n+2)g_{13}\\ ng_{12}+g_{33} & (n+2)g_{22} & (n+2)g_{23}\\ (n+2)g_{13} & (n+2)g_{23} & 2g_{12}+(n+1)g_{33}\end{pmatrix}_{g=g^*}. \tag{17}$$

(ii) They may be of type $a_{k,l}b_{\kappa,\lambda}$ with a and b symmetric in the indices, and $b_{\kappa,\kappa} = 0$. They yield cross-over exponents which are obtained from the eigenvalues of the 3×3 matrix

$$\gamma_{\mathrm{cr,s}} = \hat{d}^{(2)}, \tag{18}$$

which in one-loop order reads

$$\gamma_{\mathrm{cr,s}} = -\frac{1}{2}\begin{pmatrix} 2g_{11} & g_{33} & 2g_{13}\\ g_{33} & 2g_{22} & 2g_{23}\\ 2g_{13} & 2g_{23} & 2g_{12}+g_{33}\end{pmatrix}_{g=g^*}. \tag{19}$$

(iii) Finally they may be of type $a_{k,l}b_{\kappa,\lambda}$, but now with both a and b antisymmetric in their indices. They are obtained from

$$\gamma_{\mathrm{cr,a}} = 2\big(d^{(2)}_{12,12} - d^{(2)}_{12,21}\big), \tag{20}$$

which in one-loop order reads

$$\gamma_{\mathrm{cr,a}} = -g_{12} + \tfrac{1}{2}g_{33}. \tag{21}$$

The various γ's given here are the anomalous dimensions in terms of the length scale. The full dimension y is written as

$$y_i = D - \frac{D-2}{2} N \pm \gamma_i \tag{22}$$

for perturbations homogeneous in ϕ of order N. For γ_Φ^* the minus sign applies, whereas for the other exponents the plus sign has to be taken. The first two contributions in the last expression are the bare exponents valid for the trivial fixed point, whereas the last term constitutes the anomalous contribution. If one singles out a linear combination of the scalar products $\mathcal{I}$ as multiplied by the temperature difference τ from the critical point, then the singular part of the free energy shows the scaling behavior

$$F_{\text{sing}}(\tau, \{\mu_i\}) = |\tau|^{D\nu} F_{\text{sing},\pm}\left(\left\{\frac{\mu_i}{|\tau|^{\Delta_i}}\right\}\right) \tag{23}$$

near criticality, where τ and μ_i are multiplied by scaling operators. ν obeys $y_\tau = 1/\nu$ and the gap exponents Δ_i are related to the y_i by

$$\Delta_i = \frac{y_i}{y_\tau} = \nu y_i. \tag{24}$$

In the special case of operators bilinear in ϕ the exponents Δ_i are cross-over exponents.

3. Field rotations

One may perform a rotation between the fields ϕ_1 and ϕ_2 in the model (4),

$$\begin{pmatrix}\phi_1' \\ \phi_2'\end{pmatrix} = \begin{pmatrix}\cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi)\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2.\end{pmatrix}. \tag{25}$$

Performing the rotation (25) yields

$$\begin{pmatrix}\mathcal{I}_1' \\ \mathcal{I}_2' \\ \mathcal{I}_3'\end{pmatrix} = M \begin{pmatrix}\mathcal{I}_1 \\ \mathcal{I}_2 \\ \mathcal{I}_3\end{pmatrix}, \qquad M = \begin{pmatrix}\frac{1}{2} + \frac{1}{2}\cos(2\varphi) & \frac{1}{2} - \frac{1}{2}\cos(2\varphi) & \sqrt{\frac{1}{2}}\sin(2\varphi) \\ \frac{1}{2} - \frac{1}{2}\cos(2\varphi) & \frac{1}{2} + \frac{1}{2}\cos(2\varphi) & -\sqrt{\frac{1}{2}}\sin(2\varphi) \\ -\sqrt{\frac{1}{2}}\sin(2\varphi) & \sqrt{\frac{1}{2}}\sin(2\varphi) & \cos(2\varphi)\end{pmatrix}. \tag{26}$$

The matrix M is orthogonal and the interaction transforms according to

$$S_{\text{int}}(\phi_1', \phi_2', g') = \tfrac{1}{8}\mathcal{I}'^T g' \mathcal{I}', \qquad g' = MgM^T. \tag{27}$$

Obviously both sets of couplings describe the same critical behavior. One finds that

$$a_1 = g_{11} + g_{22} + 2g_{12}, \qquad a_2 = g_{11} + g_{22} + g_{33} \tag{28}$$

are invariant under the rotations, whereas

$$a_{31} = g_{11} - g_{22}, \qquad a_{32} = \sqrt{2}(g_{13} + g_{23}), \tag{29}$$

$$a_{41} = -g_{11} + 2g_{12} - g_{22} + 2g_{33}, \qquad a_{42} = -\sqrt{8}(g_{13} - g_{23}) \tag{30}$$

transform according to

$$\begin{pmatrix}a_{31}' \\ a_{32}'\end{pmatrix} = \begin{pmatrix}\cos(2\varphi) & \sin(2\varphi) \\ -\sin(2\varphi) & \cos(2\varphi)\end{pmatrix}\begin{pmatrix}a_{31} \\ a_{32}\end{pmatrix} \tag{31}$$

and

$$\begin{pmatrix}a_{41}' \\ a_{42}'\end{pmatrix} = \begin{pmatrix}\cos(4\varphi) & \sin(4\varphi) \\ -\sin(4\varphi) & \cos(4\varphi)\end{pmatrix}\begin{pmatrix}a_{41} \\ a_{42}\end{pmatrix}. \tag{32}$$

For the interactions invariant under $O(n)\times O(2)$ the amplitudes $a_{31}, a_{32}, a_{41}, a_{42}$ have to vanish. For all other interactions we may choose φ. We will choose it so that

$$a_{42} = 0, \qquad \text{that is} \quad g_{23} = g_{13}. \tag{33}$$

In the following section we will derive the FPs of (10) with the condition (33), from which all other fixed points can be obtained by means of the transformations (31) and (32) leaving the expressions (28) invariant.

4. The classification of the fixed points in the large *n* limit

4.1. The form of the projectors

In the large-n limit we may neglect the last term in (10). We express g in terms of the matrix p,

$$g = 4\epsilon p/(n+8). \tag{34}$$

We see that at criticality ($\beta_{ij} \equiv 0$) and in the limit $n \to \infty$ the matrix p becomes idempotent: $p = p^2$. The only eigenvalues of idempotent matrices are 0 and 1. Thus depending on the number k of eigenvalues 1 there are four types of symmetric (3×3) idempotent matrices $p^{(k)}$

$$p_{ij}^{(0)} = 0, \qquad p_{ij}^{(1)} = z_i z_j, \qquad p_{ij}^{(2)} = \delta_{ij} - z_i z_j, \qquad p_{ij}^{(3)} = \delta_{ij}; \qquad i, j = 1, 2, 3, \tag{35}$$

with the restriction

$$z_1^2 + z_2^2 + z_3^2 = 1. \tag{36}$$

Next the solution to (10) in the limit $n \to \infty$ is calculated by considering the first two orders in $1/(n+8)$ to g^*. This yields further conditions on z for the classes $p^{(1,2)}$.

4.2. The class $p^{(0)}$

This class consists of the trivial FP $g^* = 4\epsilon p^{(0)}/(n+8) = 0$ only. The stability matrix

$$\omega_{ij} = -(2\epsilon)\delta_{ij} \tag{37}$$

is diagonal as we can see from (10). All its eigenvalues are negative and the FP is unstable. This FP is exact and remains invariant under the orthogonal transformations.

4.3. The class $p^{(1)}$

Here, the ansatz

$$g_{ij}^* = \frac{4\epsilon}{(n+8)} z_i z_j + \frac{4\epsilon}{(n+8)^2} h_{ij} + O\left(\frac{1}{(n+8)^3}\right) \tag{38}$$

with the symmetric matrix h is put into the β-functions (10). We neglect the terms of higher order in $1/(n+8)$ and obtain

$$-\frac{\epsilon^2 z_i z_j}{n+8} - \frac{\epsilon^2 h_{ij}}{(n+8)^2} + \frac{\epsilon^2}{n+8}\left(z_i z_k + \frac{h_{ik}}{n+8}\right)\left(z_k z_j + \frac{h_{kj}}{n+8}\right) + \frac{\epsilon^2 c_{ij}}{(n+8)^2} \cong 0, \tag{39}$$

where

$$c_{ij} \equiv C_{ij,kl,mn} z_k z_l z_m z_n. \tag{40}$$

The equation for the terms of first order in $1/(n+8)$ gives the already known condition (36).

From the equation for the terms of second order we obtain

$$-h_{ij} + z_i(z_k h_{kj}) + z_j(z_k h_{ki}) + c_{ij} = 0. \tag{41}$$

Two of these six equations fix z, the remaining four can be used to determine h. We multiply (41) by z_i and sum over i

$$-z_i h_{ij} + z_k h_{kj} + z_j(z_k h_{ki} z_i) + z_i c_{ij} = 0. \tag{42}$$

With $c_j \equiv z_i c_{ij}$ we obtain

$$z_j(z_k h_{ki} z_i) = -c_j \tag{43}$$

or

$$z_i c_j - z_j c_i = 0. \tag{44}$$

The constants $c_{ij} = c_{ji}$ in (39) can be calculated with (10) and (11). The constants $c_i = c_{ij} z_j$ then read

$$c_1 = \left(z_3^2 - 2z_1 z_2\right)\left(7z_1^2 z_2 + 3z_2^3 + 4z_2 z_3^2 - 2z_1\left(z_2^2 + z_3^2\right)\right), \tag{45}$$

$$c_2 = \left(z_3^2 - 2z_1 z_2\right)\left(z_1\left(3z_1^2 - 2z_1 z_2 + 7z_2^2\right) + 2(2z_1 - z_2)z_3^2\right), \tag{46}$$

$$c_3 = 3z_3\left(2z_1 z_2 - z_3^2\right)\left((z_1 - z_2)^2 + 2z_3^2\right). \tag{47}$$

Two of the three equations (44) turn out to be identical. With (36) we obtain the following conditions on z:

$$\begin{aligned} &\left(1 - z_{12}^2\right)\left(4 - z_{12}^2\right)z_{12}(z_1 - z_2) = 0, \\ &\left(1 - z_{12}^2\right)\left(4 - z_{12}^2\right)z_{12} z_3^2 = 0, \\ &z_{12} := z_1 + z_2. \end{aligned} \tag{48}$$

Thus solutions are given by

$$z_{12} = 0, \pm 1, \pm 2, \pm\sqrt{2}, \tag{49}$$

where the first solutions can be read off immediately from equations (48), whereas the last pair of solutions follows from $z_1 - z_2 = 0$, $z_3 = 0$ and equation (36). This last solution describes an $O(n) \times O(2)$-invariant interaction. Due to the ansatz (38) a change of the sign of the z's does not alter the fixed point. Thus z_{12} and $-z_{12}$ yield the same class of fixed points. The interaction can be written as

$$S_{\text{int}}^{(1)} = \frac{\epsilon}{2(n+8)}(z_i \mathcal{I}_i)^2 \tag{50}$$

in this large-n limit. One realizes that the rotation (26) of $\mathcal{I}$ can be rewritten as

$$\mathcal{I}_1' + \mathcal{I}_2' = \mathcal{I}_1 + \mathcal{I}_2, \tag{51}$$

$$\begin{pmatrix} \mathcal{I}_1' - \mathcal{I}_2' \\ \sqrt{2}\mathcal{I}_3' \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) & \sin(2\varphi) \\ -\sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \begin{pmatrix} \mathcal{I}_1 - \mathcal{I}_2 \\ \sqrt{2}\mathcal{I}_3 \end{pmatrix}. \tag{52}$$

Thus z_{12} in

$$z_i \mathcal{I}_i = \tfrac{1}{2} z_{12}(\mathcal{I}_1 + \mathcal{I}_2) + \tfrac{1}{2}(z_1 - z_2)(\mathcal{I}_1 - \mathcal{I}_2) + z_3 \mathcal{I}_3 \tag{53}$$

stays constant, whereas $z_1 - z_2$ and z_3 vary under rotation with

$$(z_1 - z_2)^2 + 2z_3^2 = 2 - z_{12}^2. \tag{54}$$

Thus for $z_{12} \neq \pm\sqrt{2}$ one obtains a whole continuum of solutions.

The eigenvalues of the stability matrix are determined in appendix A. In leading order they are independent of z_{12}. Similarly one can determine the other exponents from equations (13)–(21) and obtain in the limit of large n

$$\omega = \{(2\epsilon), 0\ (2\times), -(2\epsilon)\ (3\times)\}, \qquad \gamma_\tau^* = \{-(2\epsilon), 0\ (2\times)\},$$
$$\gamma_{\mathrm{cr}}^* = \left\{\frac{(2\epsilon)}{n}\left(-1 \pm z_{12}\sqrt{2-z_{12}^2}\right), \frac{(2\epsilon)}{n}\left(1-z_{12}^2\right)\ (2\times)\right\},$$
$$\gamma_\Phi^* = \left\{\frac{(2\epsilon)^2}{8n}\left(1 \pm z_{12}\sqrt{2-z_{12}^2}\right)\right\}. \tag{55}$$

Here and in the following exponents appearing several times are indicated by $(\ldots\times)$. If a $\pm$ appears in an exponent, then exponents with both signs contribute. The exponent $\gamma_{\mathrm{cr,a}}^*$ is always the last one of γ_{cr}^*.

4.4. The class $p^{(2)}$

Here the ansatz

$$g_{ij}^* = \frac{4\epsilon}{n+8}(\delta_{ij} - z_i z_j) + \frac{4\epsilon}{(n+8)^2}h_{ij} + O\left(\frac{1}{(n+8)^3}\right), \tag{56}$$

with a symmetric matrix h is put into (10). This leads to

$$\frac{-\epsilon^2}{n+8}(\delta_{ij} - z_i z_j) - \frac{\epsilon^2 h_{ij}}{(n+8)^2} + \frac{\epsilon^2}{n+8}\left(\delta_{ik} - z_i z_k + \frac{h_{ik}}{n+8}\right)\left(\delta_{kj} - z_k z_j + \frac{h_{kj}}{n+8}\right)$$
$$+ \frac{\epsilon^2 c_{ij}}{(n+8)^2} = O\left(\frac{1}{(n+8)^3}\right), \tag{57}$$

with

$$c_{ij} := C_{ij,kl,mn}(\delta_{kl} - z_k z_l)(\delta_{mn} - z_m z_n). \tag{58}$$

The equation for the first-order terms in $1/(n+8)$ gives (36) again. The equation for the second-order terms is

$$-h_{kj} z_i z_k + h_{ij} - h_{ik} z_k z_j + c_{ij} = 0. \tag{59}$$

With the same arguments which led from (41) to (48) we now deduce conditions on z corresponding to (48):

$$\left(z_{12}^2 + 1\right) z_{12}^2 (z_1 - z_2) = 0, \tag{60}$$

$$\left(z_{12}^2 + 1\right) z_{12} z_3^2 = 0. \tag{61}$$

Thus solutions are given by

$$z_{12} = 0, \pm \mathrm{i}, \pm\sqrt{2}, \tag{62}$$

where the first two solutions are immediately obvious from equations (60) and the last one follows from $z_1 = z_2$, $z_3 = 0$, and equation (36). This last solution represents an $O(n)\times O(2)$-invariant model. The interaction can be written as

$$S_{\mathrm{int}}^{(2)} = \frac{\epsilon}{2(n+8)}\left(\mathcal{I}_i\mathcal{I}_i - (z_i\mathcal{I}_i)^2\right) \tag{63}$$

in the large-n limit. Note that $\mathcal{I}_i\mathcal{I}_i$ is invariant under rotations (26). Thus the same argument concerning the invariance of z_{12} under rotations as for $p^{(1)}$ applies here. Again for $z_{12} \neq \pm\sqrt{2}$ one obtains a continuous set of models related by the transformation (51)–(54).

The stability matrix ω yields in this limit eigenvalues opposite in sign to those of $p^{(1)}$ (appendix A). Similarly one determines the other exponents from equations (13)–(21) and obtains in the limit of large n

$$\begin{aligned} &\omega = \{(2\epsilon)\ (3\times), 0\ (2\times), -(2\epsilon)\}, \qquad \gamma_\tau^* = \{-(2\epsilon)\ (2\times), 0\}, \\ &\gamma_{\rm cr}^* = \left\{\frac{(2\epsilon)}{n}\big(-2+z_{12}^2\big), \frac{(2\epsilon)}{n}\Big(-1\pm\sqrt{1+2z_{12}^2-z_{12}^4}\Big), \frac{(2\epsilon)}{n}z_{12}^2\right\}, \\ &\gamma_\Phi^* = \left\{\frac{(2\epsilon)^2}{8n}\Big(2\pm z_{12}\sqrt{2-z_{12}^2}\Big)\right\}. \end{aligned} \tag{64}$$

4.5. The class $p^{(3)}$

In the large-n limit one obtains $g^* = 4\epsilon p^{(3)}/(n+8)$, which yields the exponents in leading order

$$\begin{aligned} &\omega = \{(2\epsilon)\ (6\times)\}, \qquad \gamma_\tau^* = \{-(2\epsilon)\ (3\times)\}, \\ &\gamma_{\rm cr}^* = \left\{\frac{-3(2\epsilon)}{n}, \frac{-(2\epsilon)}{n}\ (2\times), \frac{(2\epsilon)}{n}\right\}, \qquad \gamma_\Phi^* = \left\{\frac{3(2\epsilon)^2}{8n}\ (2\times)\right\}. \end{aligned} \tag{65}$$

5. Solutions for finite n

5.1. Fixed points

In order to solve equations (10) for the couplings g^* for finite n, we observe that the 'gauge' condition $a_{42} = g_{13} - g_{23} = 0$ yields

$$\beta_{11} - \beta_{22} = -\frac{1}{2}(g_{11} - g_{22})(4\epsilon - (n+8)(g_{11} + g_{22})) = 0, \tag{66}$$

$$\beta_{13} - \beta_{23} = \frac{n+8}{2}(g_{11} - g_{22})g_{13} = 0, \tag{67}$$

$$\beta_{13} = -\frac{1}{2}g_{13}(4\epsilon - (n+8)(g_{11} + g_{12} + g_{33})) = 0. \tag{68}$$

Thus we have to solve any of the two equations

$$g_{13} = 0, \qquad g_{11} = g_{22} \tag{69}$$

$$g_{13} = 0, \qquad 4\epsilon - (n+8)(g_{11} + g_{22}) = 0 \tag{70}$$

$$g_{11} = g_{22}, \qquad 4\epsilon - (n+8)(g_{11} + g_{12} + g_{33}) = 0 \tag{71}$$

together with the three equations

$$\beta_{11} = \beta_{12} = 0, \tag{72}$$

$$\beta_{33} = 0. \tag{73}$$

If $g_{13} = 0$, then β_{33} factors

$$\beta_{33} = -\tfrac{1}{2}g_{33}(4\epsilon - (n+2)g_{33} + 2g_{11} + 2g_{22} + 8g_{12}). \tag{74}$$

Then we distinguish the two cases

$$g_{33} = 0, \tag{75}$$

$$4\epsilon - (n+2)g_{33} + 2g_{11} + 2g_{22} + 8g_{12} = 0. \tag{76}$$

One obtains the following solutions from (69), (72) and (75):

$$\begin{array}{ccccccc}
g_{11} = g_{22}, & & g_{13} = g_{23} = g_{33} = 0 & & & & \\
RS & g_{11}/\epsilon & g_{12}/\epsilon & a_1/\epsilon & a_2/\epsilon & a_{41}/\epsilon & z_{12} \\
0.1 & 0 & 0 & 0 & 0 & 0 & - \\
2.1 & \frac{4}{n+8} & 0 & \frac{8}{n+8} & \frac{8}{n+8} & -\frac{8}{n+8} & 0 \\
1.3 & \frac{2}{n+4} & \frac{2}{n+4} & \frac{8}{n+4} & \frac{4}{n+4} & 0 & \pm\sqrt{2} \\
1.2 & \frac{2n}{n^2+8} & \frac{8-2n}{n^2+8} & \frac{16}{n^2+8} & \frac{4n}{n^2+8} & -\frac{8(n-2)}{n^2+8} & 0.
\end{array} \tag{77}$$

Equations (69), (72) and (76) yield for $g_{33} \neq 0$

$$\begin{array}{cccc}
g_{11} = g_{22}, & g_{13} = g_{23} = 0 & & \\
RS & g_{11}/\epsilon & g_{12}/\epsilon & g_{33}/\epsilon \\
2.1 & \frac{2}{n+8} & \frac{2}{n+8} & \frac{4}{n+8} \\
1.2 & \frac{4}{n^2+8} & \frac{4}{n^2+8} & \frac{4(n-2)}{n^2+8} \\
3.1 & \frac{3n^2-2n+24+(n-6)\sqrt{n^2-24n+48}}{n^3+4n^2-24n+144} & \frac{-n^2-6n+72+(n+6)\sqrt{n^2-24n+48}}{n^3+4n^2-24n+144} & \frac{4(n^2+n-12-3\sqrt{n^2-24n+48})}{n^3+4n^2-24n+144} \\
2.2 & \frac{3n^2-2n+24-(n-6)\sqrt{n^2-24n+48}}{n^3+4n^2-24n+144} & \frac{-n^2-6n+72-(n+6)\sqrt{n^2-24n+48}}{n^3+4n^2-24n+144} & \frac{4(n^2+n-12+3\sqrt{n^2-24n+48})}{n^3+4n^2-24n+144} \\
RS & a_1/\epsilon & a_2/\epsilon & a_{41}/\epsilon \quad z_{12} \\
2.1 & \frac{8}{n+8} & \frac{8}{n+8} & \frac{8}{n+8} \quad 0 \\
1.2 & \frac{16}{n^2+8} & \frac{4n}{n^2+8} & \frac{8(n-2)}{n^2+8} \quad 0 \\
3.1 & \frac{4(n^2-4n+48+n\sqrt{n^2-24n+48})}{n^3+4n^2-24n+144} & \frac{2(5n^2+(n-12)\sqrt{n^2-24n+48})}{n^3+4n^2-24n+144} & 0 \quad - \\
2.2 & \frac{4(n^2-4n+48-n\sqrt{n^2-24n+48})}{n^3+4n^2-24n+144} & \frac{2(5n^2-(n-12)\sqrt{n^2-24n+48})}{n^3+4n^2-24n+144} & 0 \quad \pm\sqrt{2}.
\end{array} \tag{78}$$

Equations (70), (72) and (73) with $g_{11} \neq g_{22}$ yield

$$\begin{array}{ccccc}
g_{13} = g_{23} = 0, & & & & \\
RS & g_{11,22}/\epsilon - \frac{2}{n+8} & g_{12}/\epsilon & g_{33}/\epsilon & \\
1.1 & \pm\frac{2}{n+8} & 0 & 0 & \\
1.4 & \pm\sqrt{\frac{32(1-n)}{(n+8)^3}} & \frac{6}{n+8} & 0 & \\
2.3 & \pm\sqrt{\frac{-4(3n+22)(n-2)(n+2)(n+4)(n+14)}{(n+8)^3(n^2+4n+20)^2}} & \frac{4(n+6)(n+4)}{(n+8)(n^2+4n+20)} & \frac{4(n^2-36)}{(n+8)(n^2+4n+20)} & \\
RS & a_1/\epsilon & a_2/\epsilon & a_{41}/\epsilon & z_{12} \\
1.1 & \frac{4}{n+8} & \frac{4}{n+8} & -\frac{4}{n+8} & \pm 1 \\
1.4 & \frac{16}{n+8} & \frac{4}{n+8} & \frac{8}{n+8} & \pm 2 \\
2.3 & \frac{4(3n^2+24n+68)}{(n+8)(n^2+4n+20)} & \frac{8(n+4)(n-2)}{(n+8)(n^2+4n+20)} & \frac{4(3n^2+16n-44)}{(n+8)(n^2+4n+20)} & \pm\mathrm{i}.
\end{array} \tag{79}$$

Equations (71), (72) and (73) yield for $g_{13} \neq 0$

$$g_{11} = g_{22}, \qquad g_{13} = g_{23}$$

$$\begin{array}{lllll} RS & g_{11}/\epsilon & g_{12}/\epsilon & g_{33}/\epsilon & (g_{13}/\epsilon)^2 \\ 1.1 & \frac{1}{n+8} & \frac{1}{n+8} & \frac{2}{n+8} & \frac{2}{(n+8)^2} \\ 1.4 & \frac{4}{n+8} & \frac{4}{n+8} & -\frac{4}{n+8} & \frac{16(1-n)}{(n+8)^3} \\ 2.3 & \frac{5n^2+24n-4}{(n+8)(n^2+4n+20)} & \frac{(n+10)(n+14)}{(n+8)(n^2+4n+20)} & \frac{-2(n+2)(n+14)}{(n+8)(n^2+4n+20)} & \frac{-2(3n+22)(n-2)(n+2)(n+4)(n+14)}{(n+8)^3(n^2+4n+20)^2} \\ \\ RS & a_1/\epsilon & a_2/\epsilon & a_{41}/\epsilon & z_{12} \\ 1.1 & \frac{4}{n+8} & \frac{4}{n+8} & \frac{4}{n+8} & \pm 1 \\ 1.4 & \frac{16}{n+8} & \frac{4}{n+8} & -\frac{8}{n+8} & \pm 2 \\ 2.3 & \frac{4(3n^2+24n+68)}{(n+8)(n^2+4n+20)} & \frac{8(n-2)(n+4)}{(n+8)(n^2+4n+20)} & -\frac{4(3n^2+16n-44)}{(n+8)(n^2+4n+20)} & \pm \mathrm{i} \end{array} \tag{80}$$

We consider the solutions (77)–(80) as representative solutions. They are denoted by *RS* $k.m$, where k indicates that they belong to $p^{(k)}$ in the large-n limit, and m numbers the various solutions.

There are three types of solutions:

(i) the solutions, which are invariant under $O(n)\times O(2)$. There is one solution for each k, *RS 0.1, 1.3, 2.2,* and *3.1*;

(ii) solutions for which $a_{31} = a_{32} = 0$, *RS 1.2, 2.1*;

(iii) solutions for which a's can be different from 0, *RS 1.1, 1.4,* and *2.3*. The solutions can be seen in both (79) and (80). They are obtained from one another by a rotation by $\varphi = \pi/4$.

All solutions with the exception of the trivial fixed point *RS 0.1* have an exponent $\omega = 2\epsilon$ independent of n in one-loop order, since $\beta_{ij} = -2\epsilon g_{ij}+$ term bilinear in the g's and thus $\partial\beta_{ij}/\partial g_{kl}|_{g=g^*} g^*_{kl} = 2\epsilon g^*_{ij}$.

For the solutions (i) of symmetry $O(n)\times O(2)$ equation (6) holds. Then equations (13), (17) and (19) yield the eigenvalues

$$\begin{aligned} \gamma^*_\Phi &= \left\{\frac{n+1}{4}g^{*2}_{11} + \frac{3(n-1)}{16}g^{*2}_{33} - \frac{n-1}{4}g^*_{11}g^*_{33}\ (2\times)\right\}, \\ \gamma^*_\tau &= \left\{-(n+1)g^*_{11} + \frac{n-1}{2}g^*_{33}, -g^*_{11} - \frac{n-1}{2}g^*_{33}\ (2\times)\right\}, \\ \gamma^*_{\mathrm{cr,s}} &= \left\{-g^*_{11} - \frac{1}{2}g^*_{33}, -g^*_{11} + \frac{1}{2}g^*_{33}\ (2\times)\right\}. \end{aligned} \tag{81}$$

All three sets of exponents contain two degenerate exponents. The first exponent γ^*_τ yields ν, the two other ones belong to perturbations of types $\phi_1^2 - \phi_2^2$ and $\phi_1\phi_2$. Thus they yield cross-over exponents. The first cross-over exponent γ^*_{cr} belongs to operators $b_{\kappa\lambda}(\phi_{1\kappa}\phi_{1\lambda} + \phi_{2\kappa}\phi_{2\lambda})$, the two equal exponents to $b_{\kappa\lambda}(\phi_{1\kappa}\phi_{1\lambda} - \phi_{2\kappa}\phi_{2\lambda})$ and $b_{\kappa\lambda}\phi_{1\kappa}\phi_{2\lambda}$ with symmetric $b_{\kappa\lambda}$. The degeneracies are due to the $O(2)$ invariance.

All other solutions to types (ii) and (iii) can be obtained by means of field rotations as described in section 3. These solutions yield one exponent $\omega = 0$ since the field rotations create lines of fixed points. This exponent is not a true scaling exponent, but a redundant one, since the perturbation is obtained from an infinitesimal rotation between ϕ_1 and ϕ_2.

5.2. Critical exponents

In the following, we give the critical exponents of the various fixed points.

RS 0.1. This is the trivial (interaction free) fixed point. All anomalous exponents γ^* vanish

$$\gamma^*_\Phi = \{0\ (2\times)\}, \qquad \gamma^*_\tau = \{0\ (3\times)\}, \qquad \gamma^*_{\rm cr} = \{0\ (4\times)\}, \qquad \omega = \{-(2\epsilon)\ (6\times)\}. \quad (82)$$

RS 1.1. Representatives of these solutions are given in (79) and (80). The critical exponents are given by

$$\begin{aligned} &\gamma^*_\tau = \left\{-\frac{(n+2)(2\epsilon)}{n+8}, 0\ (2\times)\right\}, \qquad \gamma^*_{\rm cr} = \left\{-\frac{2(2\epsilon)}{n+8}, 0\ (3\times)\right\}, \\ &\gamma^*_\Phi = \left\{\frac{(n+2)(2\epsilon)^2}{4(n+8)^2}, 0\right\}, \qquad \omega = \left\{(2\epsilon), -(2\epsilon)\ (2\times), -\frac{(n+6)(2\epsilon)}{n+8}, -\frac{6(2\epsilon)}{n+8}, 0\right\}. \end{aligned} \quad (83)$$

RS 1.2. Representatives are given in (77) and (78). The critical exponents are

$$\begin{aligned} &\gamma^*_\Phi = \left\{\frac{n(n^2-3n+8)(2\epsilon)^2}{8(n^2+8)^2}\ (2\times)\right\}, \qquad \gamma^*_\tau = \left\{-\frac{3n(2\epsilon)}{n^2+8}, \frac{(1-n)n(2\epsilon)}{n^2+8}, \frac{(n-4)(2\epsilon)}{n^2+8}\right\}, \\ &\gamma^*_{\rm cr} = \left\{-\frac{n(2\epsilon)}{n^2+8}\ (2\times), \frac{(n-4)(2\epsilon)}{n^2+8}\ (2\times)\right\}, \\ &\omega = \left\{0, (2\epsilon), \frac{8(n-1)(2\epsilon)}{n^2+8}, \frac{(4-n)(2+n)(2\epsilon)}{n^2+8}, \right. \\ &\qquad \left. \frac{(4-n)(n-2)(2\epsilon)}{n^2+8}, \frac{(2-n)(4+n)(2\epsilon)}{n^2+8}\right\}. \end{aligned} \quad (84)$$

In the representation (77) $g_{13} = g_{23} = 0$ holds and the γ_τ matrix (17) becomes a block matrix and has the eigenvalues $\gamma^*_\tau = \{\gamma^*_{\tau_1}, \gamma^*_{\tau_2}, \gamma^*_{\tau_3}\}$ which in the case of $g^*_{11} = g^*_{22}$ belong to the eigenvectors $(1, 1, 0)$, $(1, -1, 0)$ and $(0, 0, 1)$ respectively in our convention. Thus the first entry represents an ordinary critical exponent when $\tau_1 = \tau_2$, the third entry is the critical exponent of τ_3, and the second entry as well as the exponents $\gamma^*_{\rm cr}$ are related to the crossover.

RS 1.3. This solution is not only invariant under $O(n)\times O(2)$, but even under $O(2n)$. g^* is given in (77). Its critical exponents are

$$\begin{aligned} &\gamma^*_\Phi = \left\{\frac{(2n+2)(2\epsilon)^2}{4(2n+8)^2}\ (2\times)\right\}, \qquad \gamma^*_\tau = \left\{-2\frac{(2n+2)(2\epsilon)}{2n+8}, -\frac{2(2\epsilon)}{2n+8}\ (2\times)\right\}, \\ &\gamma^*_{\rm cr} = \left\{-\frac{2(2\epsilon)}{2n+8}\ (4\times)\right\}, \qquad \omega = \left\{(2\epsilon), \frac{8(2\epsilon)}{2n+8}\ (2\times), \frac{(4-2n)(2\epsilon)}{2n+8}\ (3\times)\right\}. \end{aligned} \quad (85)$$

The last two exponents of γ^*_τ belong to cross-over exponents (discussion after (81)). These exponents and all exponents $\gamma^*_{\rm cr}$ are equal.

RS 1.4. Its representative couplings are given in (79) and (80). In one-loop order one obtains the exponents

$$\gamma^*_\Phi = \left\{ \frac{(n^2+37n+16)(2\epsilon)^2}{8(n+8)^3} \pm (n+2)\frac{\sqrt{2(1-n)}(2\epsilon)^2}{2(n+8)^{5/2}} \right\},$$
$$\gamma^*_\tau = \left\{ -\frac{(2+n)(2\epsilon)}{2(n+8)} \pm \frac{\sqrt{n^3+48n^2+32}(2\epsilon)}{2(n+8)^{3/2}}, -\frac{3(2\epsilon)}{n+8} \right\},$$
$$\gamma^*_{cr} = \left\{ -\frac{(2\epsilon)}{n+8} \pm \frac{2\sqrt{2(1-n)}(2\epsilon)}{(n+8)^{3/2}}, -\frac{3(2\epsilon)}{n+8}\ (2\times) \right\},$$
$$\omega = \left\{ 0, (2\epsilon), \frac{(6-n)(2\epsilon)}{n+8}, \frac{(10-n)(2\epsilon)}{n+8}, -\frac{(n+2)(2\epsilon)}{2(n+8)} \pm \frac{\sqrt{n^2-188n+196}(2\epsilon)}{2(n+8)} \right\}. \tag{86}$$

We consider the coupling in two-loop order, since it yields in order ϵ the region in which the couplings are real. Using the representation (79) the couplings may be written as

$$g^*_{11,22} = u^{(1)}\epsilon + u^{(2)}\epsilon^2 \pm V + O(\epsilon^3), \tag{87}$$

$$V = v^{(1)}\sqrt{w}\epsilon + \frac{v^{(2)}}{\sqrt{w}}\epsilon^2, \tag{88}$$

with

$$u^{(1)} = \frac{2}{n+8},$$
$$u^{(2)} = -\frac{2(n^3+24n^2-27n-160)}{(n+8)^4},$$
$$v^{(1)} = \frac{4}{(n+8)^2},$$
$$v^{(2)} = \frac{n^4+80n^3-2004n^2-880n+2560}{2(n+8)^4}, \tag{89}$$
$$w = 2(1-n)(8+n),$$
$$g^*_{12} = \frac{6}{n+8}\epsilon + \frac{-n^3-66n^2+450n+832}{(n+8)^4}\epsilon^2,$$
$$g^*_{13} = g^*_{23} = g^*_{33} = 0.$$

Now V can be rewritten as

$$V = v^{(1)}\epsilon\sqrt{w + \frac{2v^{(2)}}{v^{(1)}}\epsilon}. \tag{90}$$

Thus with $w(n_0) = 0$ the limit of real couplings is given by

$$n_c = n_0 - \frac{2v^{(2)}(n_0)}{v^{(1)}(n_0)w'(n_0)}\epsilon, \tag{91}$$

which in our case yields $n_c = 1 - (2\epsilon)/48 + O(2\epsilon)^2$.

RS 2.1. Representatives in one-loop order are given in (77) and (78). Two of the exponents ω equal 0 for any n in one-loop order; one is due to the invariance under rotations between the fields ϕ. The other one indicates that there may branch off a second line of FPs. Indeed one finds besides the FP of two decoupled systems $g^*_{11} = g^*_{22}, g^*_{12} = g^*_{33} = g^*_{13} = g^*_{23} = 0$ (which we denote *RS 2.1a*) another solution with $g^*_{11} = g^*_{22}, g^*_{12}, g^*_{33} = O(\epsilon^2), g^*_{13} = g^*_{23} = O(\epsilon^{3/2})$, which we denote *RS 2.1b*. Both types of FPs agree in one-loop order, but differ in the next order. Note that the first FP has $a_{31} = a_{32} = 0$, whereas the second does not show this symmetry. In the following we give the FPs and critical exponents in two-loop order (for γ_Φ^* in three-loop order).

First the general scheme to obtain the FPs beyond first order is explained. Let the β-function up to two-loop order read

$$\beta_i = -(2\epsilon)g_i + \sum_{pq} k_{ipq} g_p g_q + \sum_{pqr} l_{ipgr} g_p g_q g_r + \cdots, \tag{92}$$

where the indices $_i, _p, _q, _r$ replace the double indices $_{ij}$ and expand the contributions in one-loop order

$$k_{ipq} = k^0_{ipq} + \epsilon k^1_{ipq} + \cdots \tag{93}$$

and similarly the higher-loop orders. With

$$g_i = \epsilon g_{1,i} + \epsilon^2 g_{2,i} + \cdots \tag{94}$$

one obtains from $\beta_i(g^*)$ order ϵ^r, $r > 2$ the equation

$$B_{ij} g^*_{r-1,j} = \text{r.h.s} \tag{95}$$

$$B_{ij} := -2\delta_{ij} + 2\sum_p k^0_{ipq} g^*_{1,p}, \tag{96}$$

where the rhs of equation (95) contains only $g^*_{r',q}$ with $r' < r - 1$. The matrix B is the matrix ω in one-loop order. If none of the eigenvalues of this matrix vanishes, then equation (95) can be used to calculate g^*_r in increasing order r. If due to the rotation invariance one of the eigenvalues vanishes, then the condition (33) reduces the number of independent couplings by 1 and eliminates the vanishing eigenvalue. If, however, a second eigenvalue vanishes, then the calculation has to be modified. For this *RS 2.1* we assume $g_{22} = g_{11}$ and expand g_{11}, g_{12}, g_{33} as in equation (94), but denote $g_u = g_{13} = g_{23}$ and expand

$$g_u = \epsilon^{3/2} g_{1,u} + \epsilon^{5/2} g_{2,u} + \cdots. \tag{97}$$

From now on the indices $i, p, q, \ldots$ stand only for the double indices 11, 12, 33, but not for 13.

Order ϵ^2 of $\beta_i(g^*) = 0$ is fulfilled by the solutions of *RS 1.2*

$$g^*_{1,11} = \frac{4}{n+8}, \qquad g^*_{1,12} = g^*_{1,33} = 0. \tag{98}$$

Order $\epsilon^{5/2}$ of $\beta_u = 0$ yields

$$B_{uu} g^*_{1,u} = 0. \tag{99}$$

Since for the FP *RS 2.1* $B_{uu} = 0$, this is automatically fulfilled. Next $\beta_i(g^*) = 0$ in order ϵ^3 yields

$$\sum_q B_{iq} g^*_{2,q} + k^0_{iuu} g^{*2}_{1,u} + \sum_{pq} k^1_{ipq} g^*_{1,p} g^*_{1,q} + \sum_{pqr} l_{ipqr} g^*_{1,p} g^*_{1,q} g^*_{1,r} = 0, \tag{100}$$

from which one calculates $g^*_{2,i}$. Note that it depends on the yet unknown $g^{*2}_{1,u}$. Now $\beta(g^*) = 0$ in order $\epsilon^{7/2}$ yields

$$B_{uu} g^*_{2,u} + C_u g^*_{1,u} = 0, \qquad C_u := 2\sum_p k^0_{upu} g^*_{2,p} + 2\sum_p k^1_{upu} g^*_{1,p} + 3\sum_{p,q} l^0_{upqu} g^*_{1,p} g^*_{1,q}. \tag{101}$$

Since $B_{uu} = 0$, we have either $g^*_{1,u} = 0$ (*RS 2.1a*) or $C_u = 0$, which constitutes a quadratic equation in $g^*_{1,u}$ yielding the FP (*RS 2.1b*).

Higher orders in ϵ are determined uniquely. Order ϵ^r, $r > 3$ of $\beta_i = 0$ yields

$$\sum_q B_{iq} g^*_{r-1,q} + 2k^0_{1uu} g^*_{1,u} g^*_{r-2,u} = \text{rhs}, \tag{102}$$

where the right-hand side of the equation contains $g^*_{r',q}$ with $r' < r-1$ and $g^*_{r',u}$ with $r' < r-2$. Order $\epsilon^{r+1/2}$ of $\beta_u = 0$ yields

$$B_{uu} g^*_{r-1,u} + C_u g^*_{r-2,u} + 2\sum_p k^0_{upu} g^*_{1,u} g^*_{r-1,p} = \text{rhs}. \tag{103}$$

The rhs contains $g^*_{r',q}$ with $r' < r-1$ and $g^*_{r',u}$ with $r' < r-2$. In all cases $B_{uu} = 0$. For *RS 2.1a* one has $g^*_{1,u} = 0$ and $C_u \neq 0$, which allows a unique determination of $g^*_{r-2,u}$. Since each term of the rhs contains at least one factor $g_{r',u}$, one obtains $g_{r-2,u} = 0$. For *RS 2.1b* both $B_u = C_u = 0$ vanish. However the sum $\sum_p k^0_{upu} g^*_{1,u} g^*_{r-1,p}$ depends via $g^*_{r-1,p}$ on $g^*_{r-2,u}$. As a result one obtains from this equation $g^*_{r-2,u}$.

RS 2.1a

$$\begin{aligned} g^*_{11} &= g^*_{22} = \frac{4}{n+8}\epsilon - \frac{4(n^2-2n-20)}{(n+8)^3}\epsilon^2, \\ g^*_{12} &= g^*_{33} = g^*_{13} = g^*_{23} = 0. \end{aligned} \tag{104}$$

This solution describes two independent $O(n)$ models:

$$\begin{aligned} \gamma^*_\Phi &= \left\{ \frac{(n+2)}{4(n+8)^2}(2\epsilon)^2 - \frac{(n+2)(n^2-56n-272)}{16(n+8)^4}(2\epsilon)^3\ (2\times) \right\}, \\ \gamma^*_\tau &= \left\{ -\frac{n+2}{2(n+8)^2}(2\epsilon)^2, -\frac{n+2}{n+8}(2\epsilon) - \frac{(n+2)(13n+44)}{2(n+8)^3}(2\epsilon)^2\ (2\times) \right\}, \\ \gamma^*_{\text{cr}} &= \left\{ -\frac{2}{n+8}(2\epsilon) + \frac{(n+4)(n-22)}{2(n+8)^3}(2\epsilon)^2\ (2\times), -\frac{n+2}{2(n+8)^2}(2\epsilon)^2\ (2\times) \right\}, \\ \omega &= \left\{ (2\epsilon) - \frac{3(3n+14)}{(n+8)^2}(2\epsilon)^2\ (2\times), \frac{n-4}{n+8}(2\epsilon) + \frac{(n+2)(13n+44)}{(n+8)^3}(2\epsilon)^2, \right. \\ &\quad \left. -\frac{n+4}{n+8}(2\epsilon) - \frac{(n+4)(n-22)}{(n+8)^3}(2\epsilon)^2, \frac{n+2}{2(n+8)^2}(2\epsilon)^2, 0 \right\}. \end{aligned} \tag{105}$$

RS 2.1b. The second FP to *RS 2.1* is given by

$$\begin{aligned} g^*_{11} &= g^*_{22} = \frac{4}{n+8}\epsilon - \frac{9n^3+98n^2-400n-2272}{2(n+8)^3(n+14)}\epsilon^2, \\ g^*_{13} &= g^*_{23} = \pm\frac{\sqrt{2(n+4)(n+2)(n-4)}}{(n+8)^2\sqrt{n+14}}\epsilon^{3/2}, \\ g^*_{12} &= -\frac{n+2}{2(n+8)(n+14)}\epsilon^2, \\ g^*_{33} &= \frac{(n+2)(n-4)}{(n+8)^2(n+14)}\epsilon^2. \end{aligned} \tag{106}$$

In the limit $D = 4$ it is real for $n \geqslant 4$. Its critical exponents are

$$\begin{aligned}
\gamma_\Phi^* &= \left\{ \frac{(n+2)}{4(n+8)^2}(2\epsilon)^2 \pm \frac{(n+2)\sqrt{2(n-4)(n+2)(n+4)}}{16(n+8)^3\sqrt{n+14}}(2\epsilon)^{5/2} \right.\\
&\quad \left. - \frac{(n+2)(n^2-56n-272)}{16(n+8)^4}(2\epsilon)^3 \right\},\\
\gamma_\tau^* &= \left\{ -\frac{n+2}{(n+8)}(2\epsilon) - \frac{(n+2)(29n^2+470n+1256)}{4(n+14)(n+8)^3}(2\epsilon)^2, \right.\\
&\quad - \frac{n+2}{n+8}(2\epsilon) - \frac{(n+2)(23n^2+434n+1208)}{4(n+8)^3(n+14)}(2\epsilon)^2,\\
&\quad \left. - \frac{3(n+2)(n^2+10n+64)}{4(n+8)^3(n+14)}(2\epsilon)^2 \right\},\\
\gamma_{\mathrm{cr}}^* &= \left\{ -\frac{2}{n+8}(2\epsilon) + \frac{n^3-12n^2-660n-2416}{4(n+8)^3(n+14)}(2\epsilon)^2, \right.\\
&\quad - \frac{2}{n+8}(2\epsilon) + \frac{3n^3-4n^2-700n-2512}{4(n+8)^3(n+14)}(2\epsilon)^2,\\
&\quad \left. - \frac{(n+2)(n+6)(n+32)}{4(n+8)^3(n+14)}(2\epsilon)^2, - \frac{(n+2)(n+26)}{4(n+8)^2(n+14)}(2\epsilon)^2 \right\},\\
\omega &= \left\{ (2\epsilon) - \frac{3(3n+14)}{(n+8)^2}(2\epsilon)^2\ (2\times), - \frac{n+2}{(n+8)^2}(2\epsilon)^2, 0, \ \frac{n-4}{n+8}(2\epsilon) \right.\\
&\quad + \frac{(n+2)(15n^3+242n^2+656n+32)}{n(n+8)^3(n+14)}(2\epsilon)^2,\\
&\quad \left. - \frac{n+4}{n+8}(2\epsilon) - \frac{3n^4+12n^3-332n^2-1252n+64}{n(n+8)^3(n+14)}(2\epsilon)^2 \right\}.
\end{aligned} \tag{107}$$

RS 2.3. The representatives of this fixed point are given in (79) and (80). Its critical exponents are

$$\begin{aligned}
\gamma_\Phi^* &= \left\{ \frac{(2n^6+37n^5+348n^4+2360n^3+9376n^2+13904n-9152)(2\epsilon)^2}{8(n+8)^3(n^2+4n+20)^2} \right.\\
&\quad \left. \pm \frac{(n+2)\sqrt{-(3n+22)(n-2)(n+2)(n+4)(n+14)}(2\epsilon)^2}{8(n+8)^{5/2}(n^2+4n+20)} \right\},\\
\gamma_\tau^* &= \left\{ -\frac{(2\epsilon)(n-1)(n-2)(n+6)}{(n+8)(n^2+4n+20)}, -\frac{(n+2)(2\epsilon)}{2(n+8)} \right.\\
&\quad \left. \pm \frac{(2\epsilon)\sqrt{n^7+32n^6+512n^5+3792n^4+10064n^3-3548n^2-21376n+61184}}{2(n+8)^{3/2}(n^2+4n+20)} \right\},\\
\gamma_{\mathrm{cr}}^* &= \left\{ -\frac{(2\epsilon)}{n+8} \pm \frac{\sqrt{-2(n^5+34n^4+312n^3+752n^2-1776n-7648)}(2\epsilon)}{(n+8)^{3/2}(n^2+4n+20)}, \right.\\
&\quad \left. -\frac{(n+6)(3n+2)(2\epsilon)}{(n+8)(n^2+4n+20)}, -\frac{(n+6)(n+14)(2\epsilon)}{(n+8)(n^2+4n+20)} \right\},\\
\omega &= \left\{ 0, (2\epsilon), \frac{(2\epsilon)(n^3+10n^2-4n-232)}{(n+8)(n^2+4n+20)}, \frac{(2\epsilon)\lambda'}{2(n+8)(n^2+4n+20)} \right\},
\end{aligned} \tag{108}$$

where λ' is solution of the equation

$$\lambda'^3 + 16(n^2 + 4n + 20)\lambda'^2 - 4(n + 4)(n^5 - 18n^4 - 392n^3 - 1648n^2 - 496n + 8928)\lambda' \\ - 16(3n + 22)(n - 2)(n + 6)(n - 6)(n + 4)(n + 2)(n + 14)^2 = 0. \tag{109}$$

In an expansion in $1/(n + 8)$ one obtains the ω's

$$\begin{aligned} &(2\epsilon)\left(-\frac{6}{n+8} - \frac{296}{(n+8)^2} - \frac{11\,272}{(n+8)^3} + O\left(\frac{1}{(n+8)^4}\right)\right) \\ &(2\epsilon)\left(1 - \frac{20}{n+8} - \frac{78}{(n+8)^2} + \frac{906}{(n+8)^3} + O\left(\frac{1}{(n+8)^4}\right)\right), \\ &(2\epsilon)\left(-1 + \frac{18}{n+8} + \frac{374}{(n+8)^2} + \frac{10\,366}{(n+8)^3} + O\left(\frac{1}{(n+8)^4}\right)\right). \end{aligned} \tag{110}$$

Similarly as for *RS 1.4* we consider the coupling in two-loop order, since it yields in order ϵ the region in which the couplings are real. Using the representation (79) the couplings may be written in the form (87) and (88) with

$$\begin{aligned} u^{(1)} &= \frac{2}{n+8}, \\ u^{(2)} &= -\frac{2(n^7 + 21n^6 + 249n^5 + 1564n^4 + 2312n^3 - 13\,808n^2 - 53\,104n - 55\,360)}{(n^2+4n+20)^2(n+8)^4}, \\ v^{(1)} &= \frac{2}{(n+8)^2(n^2+4n+20)}, \\ v^{(2)} &= \frac{-2(n+2)(n+4)\begin{pmatrix} 3n^{11}+13n^{10}-2301n^9-41\,840n^8-134\,712n^7+2573\,392n^6+26\,618\,112n^5 \\ +82\,530\,752n^4-6879\,104n^3-368\,123\,392n^2-274\,477\,824n-126\,516\,224 \end{pmatrix}}{(n+14)(n+8)^4(n^2+4n+20)^3}, \\ w &= -(3n+22)(n-2)(n+2)(n+4)(n+8)(n+14), \\ g_{12}^* &= \frac{4(n+4)(n+6)}{(n+8)(n^2+4n+20)}\epsilon + \frac{2\begin{pmatrix} n^{10}-21n^9-1596n^8-24\,396n^7-124\,064n^6+251\,792n^5 \\ +5029\,824n^4+19\,095\,232n^3+27\,139\,840n^2+6788\,096n-8542\,208 \end{pmatrix}}{(n+14)(n+8)^4(n^2+4n+20)^3}\epsilon^2, \\ g_{33}^* &= \frac{4(n^2-36)}{(n+8)(n^2+4n+20)}\epsilon - \frac{4\begin{pmatrix} n^{10}+30n^9-99n^8-13\,222n^7-189\,636n^6-1087\,512n^5 \\ -1638\,768n^4+8148\,960n^3+31\,543\,872n^2+18\,656\,640n-30\,614\,016 \end{pmatrix}}{(n+14)(n+8)^4(n^2+4n+20)^3}\epsilon^2, \\ g_{13}^* &= g_{23}^* = 0. \end{aligned} \tag{111}$$

From (90) and (91) we obtain $n_c = 2 - (2\epsilon)/140 + O(2\epsilon)^2$.

RS 2.2 and 3.1. These two fixed points are solutions of one and the same quadratic equation. Both fixed points are $O(n)\times O(2)$ invariant. In two-loop order the solutions g^* can be written as

$$\begin{aligned} g_{ij}^* &= u_{ij}^{(1)}\epsilon + u_{ij}^{(2)}\epsilon^2 + sV_{ij}, \\ V_{ij} &= v_{ij}^{(1)}\sqrt{w}\epsilon + \frac{v_{ij}^{(2)}}{\sqrt{w}}\epsilon^2, \\ g_{11}^* &= g_{22}^*, \qquad g_{13}^* = g_{23}^* = 0, \\ u_{11}^{(1)} &= \frac{3n^2 - 2n + 24}{N}, \qquad v_{11}^{(1)} = \frac{n-6}{N}, \\ u_{12}^{(1)} &= -\frac{(n+12)(n-6)}{N}, \qquad v_{12}^{(1)} = \frac{n+6}{N}, \end{aligned}$$

$$u_{33}^{(1)} = \frac{4(n-3)(n-4)}{N}, \qquad v_{33}^{(1)} = \frac{-12}{N},$$
$$N = n^3 + 4n^2 - 24n + 144,$$
$$w = n^2 - 24n + 48, \tag{112}$$

where $s = +1$ corresponds to *RS 3.1* called chiral FP, and $s = -1$ to *RS 2.2* called antichiral. Close to $D = 4$ they are real only for $n \geqslant 22$ and $n \leqslant 2$. The critical exponents read

$$\gamma_\Phi^* = \left\{ \frac{(5n^5 - 3n^4 - 16n^3 - 656n^2 + 3072n - 1152 + s(n-3)(n+4)w^{3/2})(2\epsilon)^2}{16N^2}\ (2\times) \right\},$$
$$\gamma_\tau^* = \left\{ -\frac{(n(48+n+n^2) + s(n-3)(4+n)\sqrt{w})(2\epsilon)}{2N}, \right. \tag{113}$$
$$\left. \frac{(-2n^3 - 3n^2 + 28n - 48 + 5sn\sqrt{w})(2\epsilon)}{2N}\ (2\times) \right\}.$$

The exponent $\gamma_{\tau_1}^*$ determines ν, whereas the two degenerate ones yield cross-over exponents.

The other cross-over exponents are obtained from

$$\gamma_{\rm cr}^* = \left\{ \frac{(-5n^2 - s(n-12)\sqrt{w})(2\epsilon)}{2N}, \frac{(-n^2 + 4n - 48 - sn\sqrt{w})(2\epsilon)}{2N}\ (2\times), \right.$$
$$\left. \frac{(3n^2 + 8n - 96 - s(n+12)\sqrt{w})(2\epsilon)}{2N} \right\}. \tag{114}$$

The six exponents ω are

$$\omega = \left\{ \frac{(n+4)\big((n+4)(n-3) - 3s\sqrt{w}\big)(2\epsilon)}{N}\ (2\times), \right. \tag{115}$$

$$\frac{(n^3 + 14n^2 + 56n - 96 + s(n+8)(n-6)\sqrt{w})(2\epsilon)}{2N}\ (2\times),$$
$$\left. \frac{(-3(n^2 - 24n + 48) + s(n+4)(n-3)\sqrt{w})(2\epsilon)}{N}, (2\epsilon) \right\}. \tag{116}$$

In two-loop order one can write

$$u_{11}^{(2)} = \frac{1}{N^3}(-3n^8 + 10n^7 - 432n^6 + 1710n^5 + 7480n^4$$
$$+ 20\,976n^3 + 3456n^2 - 411\,264n + 456\,192),$$
$$u_{12}^{(2)} = \frac{1}{N^3}(n^8 + 20n^7 - 286n^6 - 3550n^5 - 11\,960n^4$$
$$+ 32\,208n^3 + 165\,888n^2 + 148\,608n - 290\,304),$$
$$u_{33}^{(2)} = -\frac{2(n-3)(n+4)}{N^3}(2n^6 + 3n^5 + 94n^4 - 2688n^3$$
$$- 5904n^2 - 20\,736n + 31\,104),$$
$$v_{ij}^{(2)} = \frac{F_{ij}w}{N^3} + \frac{v_{ij}^{(1)}L}{N^2},$$
$$F_{11} = -n^7 + 50n^6 + 552n^5 + 11\,726n^4 + 230\,912n^3 + 5022\,864n^2$$
$$+ 109\,907\,904n - 910\,069\,632,$$
$$F_{12} = -n^7 + 4n^6 + 658n^5 + 19\,546n^4 + 416\,192n^3 + 8896\,560n^2$$
$$+ 193\,441\,728n + 909\,986\,688,$$

$$F_{33} = 46n^6 - 106n^5 - 7820n^4 - 185\,280n^3 - 3873\,696n^2 \\ - 83\,533\,824n - 1820\,056\,320,$$

$$L = 11\,520\,000(287n - 632). \tag{117}$$

As a result one obtains

$$n_c = n_0 - \frac{2L(n_0)}{N^2(n_0)w'(n_0)}\epsilon \tag{118}$$

which yields $n_c = 12 \pm 4\sqrt{6} - (12 \pm 14\sqrt{6}/3)(2\epsilon) + O(2\epsilon)^2$ in agreement with [12].[3]

The FP (112) is stable for large n, where the sign s in front of the root $\sqrt{w}$ is chosen positive. The stability of this FP in three dimensions is discussed on the basis of various calculation schemes in section 11.5.3 of [4], see also [16–21, 30–32].

The large-n expansion of critical exponents for the FP (112) was performed in [12, 34]. We mention the results for the exponents γ^*_Φ and γ^*_τ in arbitrary dimension D and in the first order of $1/n$

$$\eta = \frac{6\Gamma(D-2)\sin\left(\frac{D\pi}{2}\right)}{\pi\Gamma(D/2-2)\Gamma(1+D/2)n} \tag{119}$$

$$1/\nu = D - 2 + \frac{2(2-D)(1-D)\eta}{4-D}, \qquad 1/\nu_2 = D - 2 + \frac{2(2-D)(3-2D)\eta}{3(4-D)}. \tag{120}$$

The exponents η and ν were already given in [12], the exponent ν_2 in [34], where $1/\nu_2 = 2 + \gamma^*_{\tau,2}$. It yields a cross-over exponent $\Delta = \nu/\nu_2$, compare equations (23) and (24).

5.3. Stability in 3.99 dimensions

In this subsection a number of criteria for stability of the fixed points are summarized. Since the calculations are done in first order and in some cases in second order in ϵ, they apply only to dimensions close below 4.

In many cases one can easily read of the sign of the ω's from the expressions given above. In a number of cases it is not so easy. For these cases (the last two of *RS 1.4*, the last four of *RS 2.3* and all of *RS 2.2* and *RS 3.1*) we show in table 1 the regions of n where ω is positive, negative or even complex. Table 2 gives the number of ω's with negative real part for the various FPs as function of n. Only the first order in ϵ is taken into account with the exception of that ω in *RS 2.1a/b* which is identical to zero in first order and different from zero in second order due to the splitting of the fixed points.

In table 3 we show the regions of stable couplings g. We call couplings stable if the interaction S_{int} (or its real part) is positive for any $\phi_1^2 + \phi_2^2 = 1$. If S_{int} is positive for some directions (ϕ_1, ϕ_2) but zero for special directions, then it is indicated as semistable (is). If it vanishes in all directions (*RS 0.1*), it is denoted by i. Otherwise the notation u for unstable is used.

The stability is determined for representations with $g_{13} = g_{23} = 0$. In one-loop order there is such a representation for all fixed points in (77)–(79). For $g_{11} = g_{22}$ one obtains

$$g_{11}\left(\phi_1^2\right)^2 + g_{22}\left(\phi_2^2\right)^2 + 2g_{12}\phi_1^2\phi_2^2 + 2g_{33}(\phi_1\phi_2)^2 = \left(\left(\phi_1^2\right)^2 + \left(\phi_2^2\right)^2\right)(g_{11} + q(g_{12} + q'g_{33}))$$

with $2\phi_1^2\phi_2^2 = q\left(\left(\phi_1^2\right)^2 + \left(\phi_2^2\right)^2\right), 0 \leqslant q \leqslant 1$ since $\left(\phi_1^2 - \phi_2^2\right)^2 \geqslant 0$, and $(\phi_1\phi_2)^2 = q'\phi_1^2\phi_2^2$, $0 \leqslant q' \leqslant 1$. Thus, the conditions for stability are

$$g_{11} > 0, \qquad g_{11} + g_{12} > 0, \qquad g_{11} + g_{12} + g_{33} > 0.$$

[3] Equation (4.5) in [9] is misprinted. The correct result is found in (3.10) of [12].

Table 1. Regions of n where ω is either positive ($\stackrel{+}{\to}$), negative ($\stackrel{-}{\to}$), complex with positive real part ($\stackrel{+i}{\to}$) or complex with negative real part ($\stackrel{-i}{\to}$). It is not shown if $\omega = 0$ only at a discrete value of n.

RS	Mult.	From $\stackrel{\text{sign}}{\to}$ down to
1.4	1	$\infty \stackrel{-}{\to} 186.95 \stackrel{-i}{\to} 1.05 \stackrel{-}{\to} 1 \stackrel{+}{\to} -8 \stackrel{-}{\to} -\infty$
	1	$\infty \stackrel{-}{\to} 186.95 \stackrel{-i}{\to} 1.05 \stackrel{-}{\to} -8 \stackrel{+}{\to} -\infty$
2.3	1	$\infty \stackrel{+}{\to} 4.19 \stackrel{-}{\to} -8 \stackrel{+}{\to} -\infty$
	1	$\infty \stackrel{+}{\to} 6 \stackrel{-}{\to} -8 \stackrel{+}{\to} -\infty$
	1	$\infty \stackrel{-}{\to} 55.54 \stackrel{-i}{\to} 2.02 \stackrel{-}{\to} -2 \stackrel{+}{\to} -2.17 \stackrel{+i}{\to} -4 \stackrel{-}{\to} -8 \stackrel{+}{\to} -\infty$
	1	$\infty \stackrel{-}{\to} 55.54 \stackrel{-i}{\to} 2.02 \stackrel{-}{\to} 2 \stackrel{+}{\to} -2.17 \stackrel{+i}{\to} -4 \stackrel{+}{\to} -6 \stackrel{-}{\to} -7.33 \stackrel{+}{\to} -8 \stackrel{-}{\to} -\infty$
2.2	2	$\infty \stackrel{+}{\to} 21.80 \stackrel{+i}{\to} 3 \stackrel{-i}{\to} 2.20 \stackrel{-}{\to} 2 \stackrel{+}{\to} -4 \stackrel{-}{\to} -8.68 \stackrel{+}{\to} -\infty$
	2	$\infty \stackrel{+}{\to} 21.80 \stackrel{+i}{\to} 2.20 \stackrel{+}{\to} -7.33 \stackrel{-}{\to} -8.68 \stackrel{+}{\to} -\infty$
	2	$\infty \stackrel{-}{\to} 21.80 \stackrel{+i}{\to} 2.20 \stackrel{+}{\to} 2 \stackrel{-}{\to} -8.68 \stackrel{+}{\to} -\infty$
3.1	2	$\infty \stackrel{+}{\to} 21.80 \stackrel{+i}{\to} 3 \stackrel{-i}{\to} 2.20 \stackrel{-}{\to} -4 \stackrel{+}{\to} -\infty$
	2	$\infty \stackrel{+}{\to} 21.80 \stackrel{+i}{\to} 2.20 \stackrel{+}{\to} 2 \stackrel{-}{\to} -\infty$
	2	$\infty \stackrel{+}{\to} 21.80 \stackrel{+i}{\to} 2.20 \stackrel{-}{\to} -\infty$

Table 2. Number of ω's with negative real part given as function of n.

RS	From $\stackrel{\#\text{neg.}\omega\text{s}}{\to}$ down to
0.1	$\infty \stackrel{6}{\to} -\infty$
1.1	$\infty \stackrel{4}{\to} -6 \stackrel{3}{\to} -\infty$
1.2	$\infty \stackrel{3}{\to} 4 \stackrel{1}{\to} 1 \stackrel{2}{\to} -2 \stackrel{3}{\to} -4 \stackrel{4}{\to} -\infty$
1.3	$\infty \stackrel{3}{\to} 2 \stackrel{0}{\to} -4 \stackrel{5}{\to} -\infty$
1.4	$\infty \stackrel{4}{\to} 10 \stackrel{3}{\to} 6 \stackrel{2}{\to} 1 \stackrel{1}{\to} -8 \stackrel{3}{\to} -\infty$
2.1a	$\infty \stackrel{1}{\to} 4 \stackrel{2}{\to} -2 \stackrel{3}{\to} -4 \stackrel{2}{\to} -\infty$
2.1b	$\infty \stackrel{2}{\to} 4 \stackrel{3}{\to} -2 \stackrel{2}{\to} -4 \stackrel{1}{\to} -\infty$
2.3	$\infty \stackrel{2}{\to} 6 \stackrel{3}{\to} 4.19 \stackrel{4}{\to} 2 \stackrel{3}{\to} -2 \stackrel{2}{\to} -4 \stackrel{3}{\to} 6 \stackrel{4}{\to} -7.33 \stackrel{3}{\to} -8 \stackrel{1}{\to} -\infty$
2.2	$\infty \stackrel{2}{\to} 21.80 \stackrel{0}{\to} 3 \stackrel{2}{\to} -4 \stackrel{4}{\to} -7.33 \stackrel{6}{\to} -8.68 \stackrel{0}{\to} -\infty$
3.1	$\infty \stackrel{0}{\to} 3 \stackrel{2}{\to} 2.20 \stackrel{4}{\to} 2 \stackrel{6}{\to} -4 \stackrel{4}{\to} -\infty$

If $g_{11} \neq g_{22}$, one may set $\phi_1^2 = w\phi_1'^2$ and $\phi_2^2 = \phi_2'^2 / w$ with $w = \sqrt[4]{g_{22}/g_{11}}$. Then g_{11} and g_{22} are replaced by $\sqrt{g_{11}g_{22}}$. In that case the conditions of stability read

$$g_{11} > 0, \qquad g_{22} > 0, \qquad \sqrt{g_{11}g_{22}} + g_{12} > 0, \qquad \sqrt{g_{11}g_{22}} + g_{12} + g_{33} > 0$$

If couplings are complex, they have to be replaced by their real parts.

The distinction between stable and unstable no longer applies for negative even $n = -2r$ if the fields are expressed by r pairs of anticommuting (Grassmannian) components instead, whose integration is not subject to convergency conditions as in the case of real components. (The integral over functions of scalar products of vectors with L real components and r pairs of

Table 3. Regions of stable, unstable and indifferent ϕ^4 interactions. Notations: *i* indifferent, *s* stable, *u* unstable and *c* complex couplings. The stability of *RS 2.1b* is determined in one-loop order, the regions of real and complex couplings in two-loop order.

RS	From $\overset{\text{stable/unstable/indifferent/complexcouplings}}{\rightarrow}$ down to
0.1	$\infty \overset{i}{\rightarrow} -\infty$
1.1	$\infty \overset{is}{\rightarrow} -8 \overset{u}{\rightarrow} -\infty$
1.2	$\infty \overset{s}{\rightarrow} 0 \overset{u}{\rightarrow} -\infty$
1.3	$\infty \overset{s}{\rightarrow} -4 \overset{u}{\rightarrow} -\infty$
1.4	$\infty \overset{sc}{\rightarrow} 1 \overset{s}{\rightarrow} 0 \overset{u}{\rightarrow} -8 \overset{uc}{\rightarrow} -\infty$
2.1a	$\infty \overset{s}{\rightarrow} -8 \overset{u}{\rightarrow} -\infty$
2.1b	$\infty \overset{s}{\rightarrow} 4 \overset{sc}{\rightarrow} -2 \overset{s}{\rightarrow} -4 \overset{sc}{\rightarrow} -8 \overset{uc}{\rightarrow} -14 \overset{u}{\rightarrow} -\infty$
2.3	$\infty \overset{sc}{\rightarrow} 2 \overset{s}{\rightarrow} 1 \overset{u}{\rightarrow} -2 \overset{uc}{\rightarrow} -4 \overset{u}{\rightarrow} -6 \overset{s}{\rightarrow} -7.33 \overset{sc}{\rightarrow} -8 \overset{u}{\rightarrow} -14 \overset{uc}{\rightarrow} -\infty$
2.2	$\infty \overset{s}{\rightarrow} 21.80 \overset{sc}{\rightarrow} 2.20 \overset{s}{\rightarrow} -8.68 \overset{u}{\rightarrow} -\infty$
3.1	$\infty \overset{s}{\rightarrow} 21.80 \overset{sc}{\rightarrow} 2.20 \overset{s}{\rightarrow} 1 \overset{u}{\rightarrow} -\infty$

anticommuting variables depends only on $L - 2r$ [37–39]. The scalar product has the property that it is invariant under unitary orthosymplectic transformations, which in the case of $r = 0$ reduces to orthogonal transformations.)

6. The well-known subcases

Here we review the FPs of our model (4) which also contain the actions (1)–(3). While the trivial Gaussian FP is unstable in all models, their stable FPs, apart from the stable FP of (3), are found unstable in the general model (4).

The nontrivial n-Heisenberg FP of the simple ϕ^4 model (1) is stable and corresponds to *RS 1.1* if the second field is neglected. The quantities (83) reduce to

$$\gamma_\Phi^* = \frac{(n+2)(2\epsilon)^2}{4(n+8)^2}, \qquad \gamma_\tau^* = -\frac{(n+2)(2\epsilon)}{n+8}, \qquad \gamma_{\text{cr}}^* = -\frac{2(2\epsilon)}{n+8}, \qquad \omega = (2\epsilon). \quad (121)$$

The models (2) and (3) are special cases of model (4). Since the number of independent couplings τ and g is less, the number of exponents $(\gamma_\Phi^*, \gamma_\tau^*, \omega)$ reduces to (2,2,3) for model (2) and to (1,1,2) for model (3). Those exponents γ_τ^* of (4), which are no longer γ_τ^* s of (2) and (3) belong now to the exponents γ_{cr}^*.

The $O(n)$+$O(n)$ model (2) with $3g_{11} = g_1, 3g_{22} = g_2, 6g_{12} = g_3, g_{13} = 0, g_{23} = 0, g_{33} = 0$ has six nontrivial FPs. Three of them are decoupled ($g_3^* = 0$) and therefore represent tetracritical rather than bicritical behavior [1]: the n-Heisenberg–Gaussian FP with $g_1^* = 6(2\epsilon)/(n+8)$ and $g_2^* = 0$, the Gaussian–n-Heisenberg FP with $g_2^* = 6(2\epsilon)/(n+8)$ and $g_1^* = 0$ and the n-Heisenberg–n-Heisenberg FP with $g_1^* = g_2^* = 6(2\epsilon)/(n+8)$. The critical exponents of the n-Heisenberg–Gaussian FP *RS 1.1* in (79) are

$$\gamma_\Phi^* = \left\{\frac{(n+2)(2\epsilon)^2}{4(n+8)^2}, 0\right\}, \qquad \gamma_\tau^* = \left\{-\frac{(n+2)(2\epsilon)}{n+8}, 0\right\},$$
$$\omega = \left\{(2\epsilon), -(2\epsilon), -\frac{6(2\epsilon)}{n+8}\right\}, \quad (122)$$

and the decoupled n-Heisenberg–n-Heisenberg FP *RS 2.1* in (77) has

$$\begin{aligned} \gamma_\Phi^* &= \left\{ \frac{(n+2)(2\epsilon)^2}{4(n+8)^2}\ (2\times) \right\}, \qquad \gamma_\tau^* = \left\{ -\frac{(n+2)(2\epsilon)}{n+8}\ (2\times) \right\}, \\ \omega &= \left\{ (2\epsilon)\ (2\times), \frac{(n-4)(2\epsilon)}{n+8} \right\}. \end{aligned} \tag{123}$$

The latter FP is clearly stable for $n > 4$.

The three remaining FPs have a nonvanishing g_3^* and therefore represent bicritical behavior. The first FP is the isotropic $2n$-Heisenberg FP [35, 36] *RS 1.3* in (77) with

$$\begin{aligned} \gamma_\Phi^* &= \left\{ \frac{(2n+2)(2\epsilon)^2}{4(2n+8)^2} \right\}, \qquad \gamma_\tau^* = \left\{ -\frac{(2n+2)(2\epsilon)}{2n+8}, -\frac{2(2\epsilon)}{2n+8} \right\}, \\ \omega &= \left\{ (2\epsilon), \frac{8(2\epsilon)}{2n+8}, -\frac{(2n-4)(2\epsilon)}{2n+8} \right\}. \end{aligned} \tag{124}$$

This FP is stable for $n < 2$. The first $\gamma_\tau^* \sim O(\epsilon)$ is the true critical exponent for τ, the second $\gamma_\tau^* \sim O(\epsilon/n)$ yields the cross-over exponent.

The second FP is the so-called biconical FP *RS 1.2* in (77). Its critical exponents are

$$\begin{aligned} \gamma_\Phi^* &= \left\{ \frac{n(n^2-3n+8)(2\epsilon)^2}{8(n^2+8)^2}\ (2\times) \right\}, \qquad \gamma_\tau^* = \left\{ \frac{(1-n)n(2\epsilon)}{n^2+8}, -\frac{3n(2\epsilon)}{n^2+8} \right\}, \\ \omega &= \left\{ (2\epsilon), \frac{8(n-1)(2\epsilon)}{8+n^2}, \frac{(4-n)(n-2)(2\epsilon)}{8+n^2} \right\}. \end{aligned} \tag{125}$$

The biconical FP is stable for $n = 3$ in our approximation.

The last FP is given by *RS 1.4* in (79) and is complex for $n > 1$. Its critical exponents are given by $\gamma_{\phi 1,2}^*, \gamma_{\tau 1,2}^*$ and $\omega_{2,5,6}$ of (86). This FP coincides with the biconical FP for $n = 1$.

The frustrated spin model (3) is invariant under $O(n)\times O(2)$. It is obtained by $\tau_2 = \tau_1, g_{11} = g_{22} = u/3, g_{12} = u/3 - v/6, g_{33} = v/6, g_{13} = g_{23} = 0$. It has four FPs: the trivial Gaussian FP *RS 0.1*, the isotropic $2n$-Heisenberg FP *RS 1.3*, and the fixed point *RS 2.2* and *RS 3.1*. γ_Φ^* is that of *RS 2.2* and *3.1*. γ_τ is $\gamma_{\tau_1}^*$ of (113), the other γ_{τ_i} and the $\gamma_{\rm cr}$ of (113) yield the cross-over exponents, and ω equals $\omega_{5,6}$ of (115).

The FPs 2.2 and 3.1 are complex for $2.2 < n < 21.8$ close to $D = 4$. This region decreases with decreasing D [9, 17]. The question of the range of stability in $D = 3$ is under debate [4, 16–21, 30–32].

7. Summary and conclusion

We considered in detail the $O(n)$-model (4) of two fields.

We gave the expressions for the β functions (10) and (11) and the matrices γ_Φ (13), γ_τ (17), $\gamma_{\rm cr,s}$ (18) and ω (12), and $\gamma_{\rm cr,a}$ (21) for the model (4) from which the critical exponents are obtained in one-loop order (for η in two-loop order).

Next we considered its properties under orthogonal transformations of the two fields. Two types of FPs emerge: four of them are invariant under $O(n)\times O(2)$. The other FPs are not invariant under $O(2)$ and yield lines of FPs. The transformation of the couplings under $O(2)$ was given.

A classification of the FPs in the large-n limit was given, before they were determined for general n. Under the numerous FPs the corresponding FPs of the well-known models were found. To our best knowledge the FPs *RS 2.1b* and *2.3* are new. *RS 2.1b* has the remarkable property that it agrees for arbitrary n with *RS 2.1a*, which describes two uncoupled systems, in one-loop order. For these FPs two of the exponents ω vanish in one-loop order. For special values of n some of the FPs coincide or yield an extra vanishing ω. This is left for further discussion.

The full description of the fixed-point structure and the values of the most essential critical exponents can be useful for analytical and numerical investigation of the critical features of the system near $D = 4$. In this way a better understanding of such interesting phenomena as inverse symmetry breaking, symmetry nonrestoration and reentrant phase transitions could be obtained. Our model generalizes the $O(n)+O(n)$ and the $O(n)\times O(2)$ model giving rise to a variety of multi-critical phenomena.

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Appendix A. Stability matrix in the large-*n* limit

The stability matrix in one-loop order is given by

$$\omega = \frac{\partial\beta}{\partial g} = -2\epsilon 1_6 + \frac{n+8}{2}\begin{pmatrix} 2g_{11} & 2g_{12} & 2g_{13} & 0 & 0 & 0 \\ g_{12} & g_{11}+g_{22} & g_{23} & g_{12} & g_{13} & 0 \\ g_{13} & g_{23} & g_{11}+g_{33} & 0 & g_{12} & g_{13} \\ 0 & 2g_{12} & 0 & 2g_{22} & 2g_{23} & 0 \\ 0 & g_{13} & g_{12} & g_{23} & g_{22}+g_{33} & g_{23} \\ 0 & 0 & 2g_{13} & 0 & 2g_{23} & 2g_{33} \end{pmatrix} \tag{A.1}$$

and yields in the large-n limit for $p^{(1)}$

$$\omega = -2\epsilon 1_6 + 2\epsilon B_1 B_2 \tag{A.2}$$

with

$$B_1 = \begin{pmatrix} 2z_1 & 0 & 0 \\ z_2 & z_1 & 0 \\ z_3 & 0 & z_1 \\ 0 & 2z_2 & 0 \\ 0 & z_3 & z_2 \\ 0 & 0 & 2z_3 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} z_1 & z_2 & z_3 & 0 & 0 & 0 \\ 0 & z_1 & 0 & z_2 & z_3 & 0 \\ 0 & 0 & z_1 & 0 & z_2 & z_3 \end{pmatrix}. \tag{A.3}$$

The 3×3-matrix B_2B_1

$$(B_2B_1)_{ij} = \delta_{ij} + z_i z_j \tag{A.4}$$

has one eigenvalue 2 and two eigenvalues 1. The matrix B_1B_2 has the same eigenvalues and in addition three eigenvalues 0. As a consequence the stability matrix ω has three eigenvalues -2ϵ, two eigenvalues 0 and one eigenvalue 2ϵ independent of z_{12}.

For $p^{(2)}$ the stability matrix reads in this limit

$$\omega = 2\epsilon 1_6 - 2\epsilon B_1 B_2 \tag{A.5}$$

and thus the eigenvalues are the negative of those of ω for $p^{(1)}$.

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